Continued Fractions

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What are continued fractions?

Simply put: continued fractions are fractions with a denominator that is a sum that contains another (continued) fraction.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

We usually work with *simple* continued fractions: the numerators are always equal to 1.

The denominators form a sequence and are all positive integers.

A few examples

$$\frac{14}{17} = 1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}} \qquad \pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \ddots}}}}$$

Other continued fractions: $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]$, and $\gamma = [2; 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1, \ldots]$.

Calculation

Consider the rational number $\frac{58}{13} \approx 4.4615$. As a first approximation, start with 4, which is the integer part: $\frac{58}{13} = 4 + \frac{6}{13}$.

The fractional part is the reciprocal of $\frac{13}{6} \approx 2.1666$. Use the integer part, 2, as an approximation for the reciprocal to obtain a second approximation of $4 + \frac{1}{2} = 4.5$.

Finally, $\frac{13}{6} = 2 + \frac{1}{6}$, and the fractional part, $\frac{1}{6}$, is its own continued fraction representation. We are done:

$$\frac{58}{13} = 4 + \cfrac{1}{2 + \cfrac{1}{6}}$$

Advantages of representation

The continued fraction representation of a real number is finite if and only if it is rational. In contrast, the decimal representation of a rational number may be infinite.

The simple continued fraction representation of an irrational number is unique.

Easy to calculate reciprocals: the numbers $[a_0; a_1, a_2, \ldots, a_n]$ and $[0; a_0, a_1, \ldots, a_n]$ are reciprocals.

Biggest advantage: successive approximations generated in finding the continued fraction representation of a number, (truncating the continued fraction representation) are in a certain sense the "best possible".

Approximation by rational numbers

The initial segments of an infinite continued fraction representation for an irrational number provide rational approximations to the number. These rational numbers are called the **convergents** of the continued fraction.

Larger convergents mean better approximations. What irrational number would thus be the most difficult to approximate rationally?

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}$$

The golden ratio

Theorem (Hurwitz):

$\forall \xi \in \mathbb{R} \setminus \mathbb{Q}$, $\exists$ infinitely many $m, n \in \mathbb{Z}$ coprime such that

$$\left| \xi - \frac{m}{n} \right| < \frac{1}{\sqrt{5} n^2}.$$

Fun fact: the convergents of φ are ratios of successive Fibonacci numbers.

Another fun fact:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$$

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Constructing a non-Borel set

As stated earlier, every irrational number has a unique representation by an infinite simple continued fraction $[a_0; a_1, a_2, \dots]$, where $a_1, a_2, \dots \in \mathbb{Z}^+$.

Let A be the set of all irrationals with the following property: there exists an infinite subsequence $a_{k_0}, a_{k_1}, \dots$ such that $a_{k_i} \mid a_{k_{i+1}}$ for all $i \in \mathbb{N}$.

Claim: A is not Borel. (I will not prove this.)

Minkowski question-mark function

If $x = [a_0; a_1, a_2, \dots]$, then

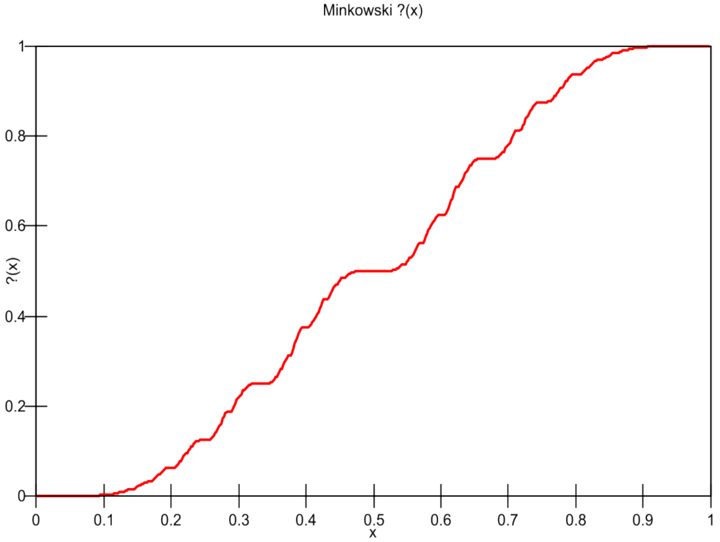
$$?(x) = a_0 + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{a_1+\cdots+a_n}}.$$

$?(x)$ is stricly increasing but has zero derivative almost everywhere.

$?(x)$ maps rationals to dyadic rationals, and quadratic irrationals to other rationals. It follows that $?(x)$ is irrational if x is either algebraic of degree > 2 or transcendental.

The graph of $?(x)$ is a type of fractal curve known as a de Rham curve.

Minkowski question-mark function



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Khinchin's constant

Theorem (Khinchin):

If $x = [a_0; a_1, a_2, \dots]$, it is almost always true that

$$\lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} = \prod_{r=1}^{\infty} \left(1 + \frac{1}{r(r+2)}\right)^{\log_2(r)} = K_0,$$

where $K_0 = 2.6854520010\dots$ (sequence A002210 in the OEIS).

Why *almost* always true? We know that rationals have finite continued fraction representations, and so obviously do not satisfy the theorem.

Also fails for quadratic irrationals (so φ) and also for e.

Proof sketch

Since the first coefficient a_0 is irrelevant, and since $\mathbb{Q}$ has Lebesgue measure zero, it suffices to study $I = \mathbb{R} \setminus \mathbb{Q} \cap (0,1)$.

Define $T : I \to I$ by $[a_1, a_2, \dots] \mapsto [a_2, a_3, \dots]$. For every Borel subset E of I, we can define the following measure:

$$\mu(E) = \frac{1}{\ln 2} \int_E \frac{dx}{1+x}.$$

Then μ is a probability measure on the σ-algebra B_I of Borel subsets of I that is equivalent to the Lebesgue measure on I and is preserved by T.

Moreover, T is an ergodic transformation on the measure space (I, B_I, μ).

Proof sketch

By the ergodic theorem, for any μ-integrable function f on I; that is, $f \in L^1(\mu)$, the average value of $f(T^k x)$ is the same for almost all x:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \int_I f \, d\mu.$$

Applying this to the function $f([a_1, a_2, \ldots]) = \ln(a_1)$, we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln(a_k) = \sum_{r=1}^{\infty} \ln\left(1 + \frac{1}{r(r+2)}\right) \log_2(r).$$

Take the exponential on both sides and we are done. □

Open problems

Although almost all numbers satisfy this property, it has not been proven for *any* real number *not* specifically constructed for the purpose.

- is it true for π?
- or $\zeta(3)$?
- or even for K_0 itself?

Is Khinchin's constant algebraic or transcendental? Is it even irrational?

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Trott constants

A Trott constant is a real number whose decimal digits are equal to the terms of its continued fraction.

$$0.10841015\cdots = 0 + \cfrac{1}{1 + \cfrac{1}{0 + \cfrac{1}{8 + \cfrac{1}{4 + \ddots}}}}$$

(A bit of cheating involved.)

Very little is known about the existence and uniqueness of such numbers. In fact, the above number was found algorithmically, using Mathematica.

Trott constants

The set of Trott constants in base b is:

- uncountable if $b = 3$ or $k^2 < b \leq k^2 + k$
- empty otherwise

That's about the only thing we know.